THE ALGEBRAIC FUNDAMENTAL GROUP OF A REDUCTIVE GROUP SCHEME OVER AN ARBITRARY BASE SCHEME

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Abstract. We define the algebraic fundamental group functor of a reductive group scheme over an arbitrary (non-empty) base scheme and prove that this functor is exact.

1. Introduction

In this paper (which is an expanded version of [Bo2]) we define the algebraic fundamental group $\pi_1(G)$ of a reductive group scheme G over an arbitrary non-empty base scheme S, thereby extending the definitions given by Merkurjev [Me], the first-named author [Bo1], Colliot-Thélène [CT] and the second-named author [GA1]. Further, we prove that the functor π_1 defined here is exact, thus generalizing results of Borovoi, Kunyavskiĭ and Gille [BKG], Colliot-Thélène [CT] and the second-named author [GA1]. We note that [GA1] makes use of flasque resolutions of reductive group schemes (introduced by Colliot-Thélène [CT] over a field), which exist only over a restricted class of base schemes. In this paper we introduce *t-resolutions* (see Definition 2.1 below), which exist over any non-empty base scheme. This enables us to work at the level of generality stated in the title.

The plan of the paper is as follows. In the remainder of this Section we introduce relevant notation. In Section 2 we use t-resolutions to define a finitely generated twisted constant S-group scheme $\pi_1(G)$ associated to any reductive S-group scheme G. In Section 3 we associate to a homomorphism $\varkappa\colon G_1 \to G_2$ of reductive S-group schemes a homomorphism $\varkappa_*\colon \pi_1(G_1) \to \pi_1(G_2)$ of finitely generated twisted constant S-group schemes and show that the resulting functor is exact. In Section 4 we apply the results of Sections 2 and 3 to abelian cohomology of reductive S-group schemes.

Notation. Throughout this paper, S is a non-empty scheme and all S-group schemes are of finite type. An S-group scheme G is called *reductive* (respectively, *semisimple*, *simply connected*) if it is affine and smooth over S and its geometric fibers are *connected* reductive (resp., semisimple, simply

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connected) algebraic groups [$\mathrm{SGA3}_{\mathrm{new}}$, Exp. XIX, Definition 2.7]. If G is a reductive group scheme over S, then $\mathrm{rad}(G)$ denotes the radical of G, i.e., the identity component of the center $Z(G)$ of G, and G^{der} denotes the derived group of G. Thus G^{der} is a normal semisimple subgroup scheme of G and $G^{\mathrm{tor}} := G/G^{\mathrm{der}}$ is the largest quotient of G which is an S-torus. We will write $\widetilde{G}$ for the simply connected central cover of G^{der} and $\mu := \mathrm{Ker}\big[\widetilde{G} \to G^{\mathrm{der}}\big]$ for the fundamental group of G^{der}. See [GA1, §2] for the existence and basic properties of $\widetilde{G}$. There exists a canonical homomorphism $\partial\colon \widetilde{G} \to G$ which factors as $\widetilde{G} \twoheadrightarrow G^{\mathrm{der}} \hookrightarrow G$. In particular, $\mathrm{Ker}\,\partial = \mu$ and $\mathrm{Coker}\,\partial = G^{\mathrm{tor}}$.

If X is a (commutative) finitely generated twisted constant S-group scheme (see [$\mathrm{SGA3}_{\mathrm{new}}$, Exp. X, Definition 5.1]), then X is quasi-isotrivial, i.e., there exists a surjective étale morphism $S' \to S$ such that $X \times_S S'$ is constant. Further, the functors

$$X \mapsto X^* := \underline{\mathrm{Hom}}_{S\text{-gr}}(X, \mathbb{G}_{m,S}) \quad \text{and} \quad M \mapsto M^* := \underline{\mathrm{Hom}}_{S\text{-gr}}(M, \mathbb{G}_{m,S})$$

are mutually quasi-inverse anti-equivalences between the categories of finitely generated twisted constant S-group schemes and S-group schemes of finite type and of multiplicative type [$\mathrm{SGA3}_{\mathrm{new}}$, Exp. X, Corollary 5.9]. Further, $M \to M^*$ and $X \to X^*$ are exact functors (see [$\mathrm{SGA3}_{\mathrm{new}}$, Exp. VIII, Theorem 3.1] and use faithfully flat descent). If G is a reductive S-group scheme, its group of characters G^* equals $(G^{\mathrm{tor}})^*$ (see [$\mathrm{SGA3}_{\mathrm{new}}$, Exp. XXII, proof of Theorem 6.2.1(i)]). Now, if T is an S-torus, the functor $\underline{\mathrm{Hom}}_{S\text{-gr}}(\mathbb{G}_{m,S}, T)$ is represented by a (free and finitely generated) twisted constant S-group scheme which is denoted by T_* and called the *group of cocharacters of* T (see [$\mathrm{SGA3}_{\mathrm{new}}$, Exp. X, Corollary 4.5 and Theorem 5.6]). There exist a canonical isomorphism of free and finitely generated twisted constant S-group schemes

$$T^* \simeq (T_*)^\vee := \mathrm{Hom}_{S\text{-gr}}(T_*, \mathbb{Z}_S). \tag{1}$$

A sequence

$$0 \to T \to H \to G \to 0 \tag{2}$$

of reductive S-group schemes and S-homomorphisms is called *exact* if it is exact as a sequence of sheaves for the fppf topology on S. In this case the sequence (2) will be called *an extension of G by T*.

If G is a reductive S-group scheme, the identity homomorphism $G \to G$ will be denoted id_G. Further, if T is an S-torus, the inversion automorphism $T \to T$ will be denoted inv_T.

2. Definition of π_1

Definition 2.1. Let G be a reductive S-group scheme. A *t-resolution of G* is a central extension

$$1 \to T \to H \to G \to 1, \tag{3}$$

where T is an S-torus and H is a reductive S-group scheme such that $H^{\rm der}$ is simply connected.

Remark 2.2. An argument similar to that given in [GA1, proof of Proposition 2.4] shows that $H^{\rm der}\cong \widetilde{G}$.

Proposition 2.3. *Every reductive S-group scheme admits a t-resolution.*

Proof. By [SGA3$_{\rm new}$, Exp. XXII, 6.2.3], the product in G defines a faithfully flat homomorphism $\mathrm{rad}(G)\times_S G^{\rm der}\to G$ which induces a faithfully flat homomorphism $\mathrm{rad}(G)\times_S \widetilde{G}\to G$. Let $\mu_1=\ker[\mathrm{rad}(G)\times_S \widetilde{G}\to G]$, which is a finite S-group scheme of multiplicative type contained in the center of $\mathrm{rad}(G)\times_S \widetilde{G}$ (see [GA1, proof of Proposition 3.2, p. 9]). By [Co, Proposition B.3.8], there exist an S-torus T and a closed immersion $\psi\colon \mu_1\hookrightarrow T$. Let H be the pushout of $\varphi\colon \mu_1\hookrightarrow \mathrm{rad}(G)\times_S \widetilde{G}$ and $\psi\colon \mu_1\hookrightarrow T$, i.e., the cokernel of the central embedding

$$(\varphi,\mathrm{inv}_T\circ\psi)_S\colon \mu_1\hookrightarrow \big(\mathrm{rad}(G)\times_S \widetilde{G}\big)\times_S T. \tag{4}$$

Then H is a reductive S-group scheme, cf. [SGA3$_{\rm new}$, Exp. XXII, Corollary 4.3.2], which fits into an exact sequence

$$1\to T\to H\to G\to 1,$$

where T is central in H. Now, as in [CT, proof of Proposition-Definition 3.1] and [GA1, proof of Proposition 3.2, p. 10], there exists an embedding of $\widetilde{G}$ into H which identifies $\widetilde{G}$ with $H^{\rm der}$. Thus $H^{\rm der}$ is simply connected, which completes the proof. □

As in [CT, p. 93] and [GA1, (3.3)], a t-resolution

$$1\to T\to H\to G\to 1 \tag{$\mathscr{R}$}$$

induces a "fundamental diagram" which, in turn, induces a canonical isomorphism in the derived category

$$\big(Z\big(\widetilde{G}\big)\xrightarrow{\partial_Z} Z(G)\big)\approx (T\to R), \tag{5}$$

where $R:=H^{\rm tor}$ (cf. [GA1, Proposition 3.4]), and a canonical exact sequence

$$1\to\mu\to T\to R\to G^{\rm tor}\to 1, \tag{6}$$

where μ is the fundamental group of $G^{\rm der}$. Since μ is finite, (6) shows that the induced homomorphism $T_*\to R_*$ is injective. Set

$$\pi_1(\mathscr{R})=\mathrm{Coker}\big[T_*\to R_*\big]. \tag{7}$$

Thus there exists an exact sequence of (étale, finitely generated) twisted constant S-group schemes

$$1\to T_*\to R_*\to \pi_1(\mathscr{R})\to 1. \tag{8}$$

Set

$$\mu(-1):=\mathrm{Hom}_{S\text{-gr}}(\mu^*,(\mathbb{Q}/\mathbb{Z})_S).$$

Proposition 2.4. *A t-resolution $\mathscr{R}$ of a reductive S-group scheme G induces an exact sequence of finitely generated twisted constant S-group schemes*

$$1 \to \mu(-1) \to \pi_1(\mathscr{R}) \to (G^{\mathrm{tor}})_* \to 1.$$

Proof. The proof is similar to the proof of [CT, Proposition 6.4], using (6). □

Definition 2.5. Let G be a reductive S-group scheme and let

$$(\mathscr{R}') \qquad 1 \to T' \to H' \to G \to 1$$

$$(\mathscr{R}) \qquad 1 \to T \to H \to G \to 1$$

be two t-resolutions of G. A *morphism from $\mathscr{R}'$ to $\mathscr{R}$*, written $\phi\colon \mathscr{R}' \to \mathscr{R}$, is a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & T' \arrow[r] \arrow[d, "\phi_T"] & H' \arrow[r] \arrow[d, "\phi_H"] & G \arrow[r] \arrow[d, "\mathrm{id}_G"] & 1 \\ 1 \arrow[r] & T \arrow[r] & H \arrow[r] & G \arrow[r] & 1, \end{tikzcd} \tag{9}$$

where ϕ_T and ϕ_H are S-homomorphisms. Note that, if $R' = (H')^{\mathrm{tor}}$ and $R = H^{\mathrm{tor}}$, then ϕ_H induces an S-homomorphism $\phi_R\colon R' \to R$.

We will say that a t-resolution $\mathscr{R}'$ of G *dominates* another t-resolution $\mathscr{R}$ of G if there exists a morphism $\mathscr{R}' \to \mathscr{R}$.

The following lemma is well-known.

Lemma 2.6. *A morphism of complexes $f\colon P \to Q$ in an abelian category is a quasi-isomorphism if and only its cone $C(f)$ is acyclic (i.e., has trivial cohomology).*

Proof. By [GM, Lemma III.3.3] there exists a short exact sequence of complexes

$$0 \to P \to \mathrm{Cyl}(f) \to C(f) \to 0, \tag{10}$$

where $\mathrm{Cyl}(f)$ is the cylinder of f. Further, the complex $\mathrm{Cyl}(f)$ is canonically isomorphic to Q in the derived category. Now the short exact sequence (10) induces a cohomology exact sequence

$$\cdots \to H^i(P) \to H^i(Q) \to H^i(C(f)) \to H^{i+1}(P) \to \dots$$

from which the lemma is immediate. □

Lemma 2.7. *Let $g\colon C \to D$ be a quasi-isomorphism of bounded complexes of split S-tori. Then the induced morphism of complexes of cocharacter S-group schemes $g_*\colon C_* \to D_*$ is a quasi-isomorphism.*

Proof. Since the assertion is local in the étale topology, we may and do assume that S is connected. The given quasi-isomorphism induces a quasi-isomorphism $g^*\colon D^* \to C^*$ of bounded complexes of free and finitely generated constant S-group schemes. Thus, by (1), it suffices to check that the functor $X \mapsto X^\vee$ on the category of bounded complexes of free and

finitely generated constant S-group schemes preserves quasi-isomorphisms. We thank Joseph Bernstein for the following argument. By Lemma 2.6 a morphism $f\colon P \to Q$ of bounded complexes in the (abelian) category of finitely generated constant S-group schemes is a quasi-isomorphism if and only if its cone $C(f)$ is acyclic. Now, if $f\colon P \to Q$ is a quasi-isomorphism and P and Q are bounded complexes of *free* and finitely generated constant S-group schemes, then $C(f)$ is an acyclic complex of free and finitely generated constant S-group schemes. We see immediately that the dual complex

$$C(f)^\vee = C\big(f^\vee\big)\big[-1\big]$$

is acyclic, whence $f^\vee$ is a quasi-isomorphism by Lemma 2.6. $\square$

Lemma 2.8. *Let G be a reductive S-group scheme and let $\mathscr{R}'$ be a t-resolution of G which dominates another t-resolution $\mathscr{R}$ of G. Then a morphism of t-resolutions $\phi\colon \mathscr{R}' \to \mathscr{R}$ induces an isomorphism of finitely generated twisted constant S-group schemes $\pi_1(\phi)\colon \pi_1(\mathscr{R}') \xrightarrow{\sim} \pi_1(\mathscr{R})$ which is independent of the choice of ϕ.*

Proof. Let $\mathscr{R}'\colon 1 \to T' \to H' \to G \to 1$ and $\mathscr{R}\colon 1 \to T \to H \to G \to 1$ be the given t-resolutions of G, as in Definition 2.5, and set $R = H^{\mathrm{tor}}$ and $R' = (H')^{\mathrm{tor}}$. Since the assertion is local in the étale topology, we may and do assume that the tori T, T', R and R' are split and that S is connected. From (6) we see that the morphism of complexes of split tori (in degrees 0 and 1)

$$(\phi_T, \phi_R)\colon \big(T' \to R'\big) \to (T \to R)$$

is a quasi-isomorphism. Now by Lemma 2.7,

$$\pi_1(\phi) := H^1((\phi_T, \phi_R)_*)\colon \pi_1(\mathscr{R}') \xrightarrow{\sim} \pi_1(\mathscr{R})$$

is an isomorphism. In order to show that this isomorphism does not depend on the choice of ϕ, assume that $\psi\colon \mathscr{R}' \to \mathscr{R}$ is another morphism of t-resolutions. It is clear from diagram (9) that ψ_H differs from ϕ_H by some homomorphism $H' \to T$ which factors through $R' = (H')^{\mathrm{tor}}$. It follows that the induced homomorphisms $(\psi_R)_*, (\phi_R)_*\colon R'_* \to R_*$ differ by a homomorphism which factors through T_*. Consequently, the induced homomorphisms

$$\pi_1(\phi), \pi_1(\psi)\colon \mathrm{Coker}\,[T'_* \to R'_*] \to \mathrm{Coker}\,[T_* \to R_*]$$

coincide. $\square$

Proposition 2.9. *Let $\varkappa\colon G_1 \to G_2$ be a homomorphism of reductive S-group schemes and let*

$$(\mathscr{R}_1) \qquad 1 \to T_1 \to H_1 \to G_1 \to 1$$

$$(\mathscr{R}_2) \qquad 1 \to T_2 \to H_2 \to G_2 \to 1$$

be t-resolutions of G_1 and G_2, respectively. Then there exists an exact commutative diagram

$$
\begin{tikzcd}
1 \arrow[r] & T_1 \arrow[r] & H_1 \arrow[r] & G_1 \arrow[r] & 1 \\
1 \arrow[r] & T_1' \arrow[r] \arrow[u] \arrow[d] & H_1' \arrow[r] \arrow[u] \arrow[d] & G_1 \arrow[r] \arrow[u, "\mathrm{id}_G"'] \arrow[d, "\varkappa"] & 1 \\
1 \arrow[r] & T_2 \arrow[r] & H_2 \arrow[r] & G_2 \arrow[r] & 1,
\end{tikzcd}
\tag{11}
$$

where the middle row is a t-resolution of G_1.

Proof. We follow an idea of Kottwitz [Ko, Proof of Lemma 2.4.4]. Let $H_1' = H_1 \times_{G_2} H_2$, where the morphism $H_1 \to G_2$ is the composition $H_1 \to G_1 \xrightarrow{\varkappa} G_2$. Clearly, there are canonical morphisms $H_1' \to H_1$ and $H_1' \to H_2$. Now, since $H_2 \to G_2$ is faithfully flat, so also is $H_1' \to H_1$. Consequently the composition $H_1' \to H_1 \to G_1$ is faithfully flat as well. Let T_1' denote its kernel, i.e., $T_1' = S \times_{G_1} H_1'$. Then

$$T_1' = (S \times_{G_1} H_1) \times_{G_2} H_2 = T_1 \times_S (S \times_{G_2} H_2) = T_1 \times_S T_2,$$

which is an S-torus. The existence of diagram (11) is now clear. Further, since T_i is central in H_i $(i = 1, 2)$, $T_1' = T_1 \times_S T_2$ is central in $H_1' = H_1 \times_{G_2} H_2$. The S-group scheme H_1' is affine and smooth over S and has connected reductive fibers, i.e., is a reductive S-group scheme. Further, the faithfully flat morphism $H_1' \to G_1$ induces a surjection $(H_1')^{\mathrm{der}} \to G_1^{\mathrm{der}}$ with (central) kernel $T_1' \cap (H_1')^{\mathrm{der}}$. Since $(H_1')^{\mathrm{der}}$ is semisimple, the last map is in fact a central isogeny. Consequently, $(H_1')^{\mathrm{der}} \to H_1^{\mathrm{der}} = \widetilde{G}_1$ is a central isogeny as well (see Remark 2.2), whence $(H_1')^{\mathrm{der}} = \widetilde{G}_1$ is simply connected. Thus the middle row of (11) is indeed a t-resolution of G_1. □

Corollary 2.10. *Let $\mathscr{R}_1$ and $\mathscr{R}_2$ be two t-resolutions of a reductive S-group scheme G. Then there exists a t-resolution $\mathscr{R}_3$ of G which dominates both $\mathscr{R}_1$ and $\mathscr{R}_2$.*

Proof. This is immediate from Proposition 2.9 (with $G_1 = G_2 = G$ and $\varkappa = \mathrm{id}_G$ there). □

Lemma 2.11. *Let $\mathscr{R}_1$ and $\mathscr{R}_2$ be two t-resolutions of a reductive S-group scheme G. Then there exists a canonical isomorphism of finitely generated twisted constant S-group schemes $\pi_1(\mathscr{R}_1) \cong \pi_1(\mathscr{R}_2)$.*

Proof. By Corollary 2.10, there exists a t-resolution $\mathscr{R}_3$ of G and morphisms of resolutions $\mathscr{R}_3 \to \mathscr{R}_1$ and $\mathscr{R}_3 \to \mathscr{R}_2$. Thus, Lemma 2.8 gives a composite isomorphism $\psi_{\mathscr{R}_3} \colon \pi_1(\mathscr{R}_1) \xrightarrow{\sim} \pi_1(\mathscr{R}_3) \xrightarrow{\sim} \pi_1(\mathscr{R}_2)$. Let $\mathscr{R}_4$ be another t-resolution of G which dominates both $\mathscr{R}_1$ and $\mathscr{R}_2$ and let $\psi_{\mathscr{R}_4} \colon \pi_1(\mathscr{R}_1) \xrightarrow{\sim} \pi_1(\mathscr{R}_4) \xrightarrow{\sim} \pi_1(\mathscr{R}_2)$ be the corresponding composite isomorphism. There exists a t-resolution $\mathscr{R}_5$ which dominates both $\mathscr{R}_3$ and $\mathscr{R}_4$.

Then $\mathscr{R}_5$ dominates $\mathscr{R}_1$ and $\mathscr{R}_2$ and we obtain a composite isomorphism $\psi_{\mathscr{R}_5}\colon \pi_1(\mathscr{R}_1) \xrightarrow{\sim} \pi_1(\mathscr{R}_5) \xrightarrow{\sim} \pi_1(\mathscr{R}_2)$. We have a diagram of t-resolutions

$$\begin{tikzcd}
& \mathscr{R}_5 \arrow[dl] \arrow[ddl] \arrow[dr] \arrow[ddr] & \\
\mathscr{R}_3 \arrow[d] \arrow[drr] & & \mathscr{R}_4 \arrow[d] \arrow[dll] \\
\mathscr{R}_1 & & \mathscr{R}_2 \,,
\end{tikzcd}$$

which may not commute. However, by Lemma 2.8, this diagram induces a *commutative* diagram of twisted constant S-group schemes and their isomorphisms

$$\begin{tikzcd}
& \pi_1(\mathscr{R}_5) \arrow[dl] \arrow[ddl] \arrow[dr] \arrow[ddr] & \\
\pi_1(\mathscr{R}_3) \arrow[d] \arrow[drr] & & \pi_1(\mathscr{R}_4) \arrow[d] \arrow[dll] \\
\pi_1(\mathscr{R}_1) & & \pi_1(\mathscr{R}_2) \,.
\end{tikzcd}$$

We conclude that

$$\psi_{\mathscr{R}_3} = \psi_{\mathscr{R}_5} = \psi_{\mathscr{R}_4}\colon \pi_1(\mathscr{R}_1) \xrightarrow{\sim} \pi_1(\mathscr{R}_2),$$

from which we deduce the existence of a *canonical* isomorphism $\psi\colon \pi_1(\mathscr{R}_1) \xrightarrow{\sim} \pi_1(\mathscr{R}_2)$. $\square$

Definition 2.12. Let G be a reductive S-group scheme. Using the preceding lemma, we will henceforth identify the S-group schemes $\pi_1(\mathscr{R})$ as $\mathscr{R}$ ranges over the family of all t-resolutions of G. Their common value will be denoted by $\pi_1(G)$ and called the *algebraic fundamental group of* G. Thus

$$\pi_1(G) = \pi_1(\mathscr{R})$$

for any t-resolution $\mathscr{R}$ of G.

Note that, by (8), a t-resolution $1 \to T \to H \to G \to 1$ of G induces an exact sequence

$$1 \to T_* \to (H^{\mathrm{tor}})_* \to \pi_1(G) \to 1. \tag{12}$$

Further, by Proposition 2.4, there exists a canonical exact sequence

$$1 \to \mu(-1) \to \pi_1(G) \to (G^{\mathrm{tor}})_* \to 1. \tag{13}$$

Remark 2.13. One can also define $\pi_1(G)$ using m-resolutions. By an *m-resolution of G* we mean a short exact sequence

$$1 \to M \to H \to G \to 1, \tag{$\mathscr{R}$}$$

where H is a reductive S-group scheme such that $H^{\,\mathrm{der}}$ is simply connected, and M is an S-group scheme of multiplicative type. Clearly, a t-resolution of G is in particular an m-resolution of G. It is very easy to see that any reductive S-group scheme G admits an m-resolution: we can take $H := \mathrm{rad}(G)\times_S \widetilde{G}$, with the homomorphism $H\to G$ from the beginning of the proof of Proposition 2.3, and set $M:=\mu_1=\mathrm{Ker}\,[H\to G]$, which is a finite S-group scheme of multiplicative type.

Now let $\mathscr{R}$ be an m-resolution of G and consider the induced homomorphism $M\to H^{\,\mathrm{tor}}$. We claim that there exists a complex of S-tori $T\to R$ which is isomorphic to $M\to H^{\,\mathrm{tor}}$ in the derived category. Indeed, by [Co, Proposition B.3.8] there exists an embedding $M\hookrightarrow T$ of M into an S-torus T. Denote by R the pushout of the homomorphisms $M\to H^{\,\mathrm{tor}}$ and $M\to T$. Then the complex of S-tori $T\to R$ is quasi-isomorphic to the complex $M\to H^{\,\mathrm{tor}}$, as claimed.

Now we choose an m-resolution $\mathscr{R}$ of G, a complex of S-tori $T\to R$ which is isomorphic to $M\to H^{\,\mathrm{tor}}$ in the derived category, and set $\pi_1(G)=\pi_1(\mathscr{R}):=\mathrm{Coker}\,[T_*\to R_*]$.

3. Functoriality and exactness of π_1

In this section we show that π_1 is an exact covariant functor from the category of reductive S-group schemes to the category of finitely generated twisted constant S-group schemes.

Definition 3.1. Let $\varkappa\colon G_1\to G_2$ be a homomorphism of reductive S-group schemes. A *t-resolution of $\varkappa$*, written $\varkappa_{\mathscr{R}}\colon \mathscr{R}_1\to\mathscr{R}_2$, is an exact commutative diagram

$$\begin{tikzcd}
(\mathscr{R}_1) & 1 \arrow[r] & T_1 \arrow[r] \arrow[d, "\varkappa_T"] & H_1 \arrow[r] \arrow[d, "\varkappa_H"] & G_1 \arrow[r] \arrow[d, "\varkappa"] & 1 \\
(\mathscr{R}_2) & 1 \arrow[r] & T_2 \arrow[r] & H_2 \arrow[r] & G_2 \arrow[r] & 1,
\end{tikzcd}$$

where $\mathscr{R}_1$ and $\mathscr{R}_2$ are t-resolutions of G_1 and G_2, respectively.

Thus, if G is a reductive S-group scheme and $\mathscr{R}'$ and $\mathscr{R}$ are two t-resolutions of G, then a morphism from $\mathscr{R}'$ to $\mathscr{R}$ (as in Definition 2.5) is a t-resolution of $\mathrm{id}_G\colon G\to G$.

Remark 3.2. A t-resolution $\varkappa_{\mathscr{R}}\colon\mathscr{R}_1\to\mathscr{R}_2$ of $\varkappa\colon G_1\to G_2$ induces a homomorphism of finitely generated twisted constant S-group schemes

$$\pi_1(\varkappa_{\mathscr{R}})\colon \pi_1(\mathscr{R}_1)\to\pi_1(\mathscr{R}_2).$$

If G_3 is a third reductive S-group scheme, $\lambda\colon G_2\to G_3$ is an S-homomorphism and $\lambda_{\mathscr{R}}\colon\mathscr{R}_2\to\mathscr{R}_3$ is a t-resolution of λ, then $\lambda_{\mathscr{R}}\circ\varkappa_{\mathscr{R}}\colon\mathscr{R}_1\to\mathscr{R}_3$ is a t-resolution of $\lambda\circ\varkappa$ and

$$\pi_1(\lambda_{\mathscr{R}}\circ\varkappa_{\mathscr{R}})=\pi_1(\lambda_{\mathscr{R}})\circ\pi_1(\varkappa_{\mathscr{R}}).$$

Lemma 3.3. *Let $\varkappa\colon G_1 \to G_2$ be a homomorphism of reductive S-group schemes and let $\mathscr{R}_2$ be a t-resolution of G_2. Then there exists a t-resolution $\varkappa_{\mathscr{R}}\colon \mathscr{R}_1 \to \mathscr{R}_2$ of $\varkappa$ for a suitable choice of t-resolution $\mathscr{R}_1$ of G_1. In particular, every homomorphism of reductive S-group schemes admits a t-resolution.*

Proof. Choose any t-resolution $\mathscr{R}_1'$ of G_1 and apply Proposition 2.9 to $\varkappa$, $\mathscr{R}_1'$ and $\mathscr{R}_2$. $\square$

Definition 3.4. Let $\varkappa\colon G_1 \to G_2$ be a homomorphism of reductive S-group schemes and let $\varkappa_{\mathscr{R}}'\colon \mathscr{R}_1' \to \mathscr{R}_2'$ and $\varkappa_{\mathscr{R}}\colon \mathscr{R}_1 \to \mathscr{R}_2$ be two t-resolutions of $\varkappa$. A *morphism from $\varkappa_{\mathscr{R}}'$ to $\varkappa_{\mathscr{R}}$*, written $\varkappa_{\mathscr{R}}' \to \varkappa_{\mathscr{R}}$, is a commutative diagram

$$\begin{array}{ccc} \mathscr{R}_1' & \xrightarrow{\varkappa_{\mathscr{R}}'} & \mathscr{R}_2' \\ \downarrow & & \downarrow \\ \mathscr{R}_1 & \xrightarrow{\varkappa_{\mathscr{R}}} & \mathscr{R}_2, \end{array}$$

where the left-hand vertical arrow is a t-resolution of id_{G_1} and the right-hand vertical arrow is a t-resolution of id_{G_2}. By a t-resolution *dominating* a t-resolution $\varkappa_{\mathscr{R}}$ of $\varkappa$ we mean a t-resolution $\varkappa_{\mathscr{R}}'$ of $\varkappa$ admitting a morphism $\varkappa_{\mathscr{R}}' \to \varkappa_{\mathscr{R}}$.

Lemma 3.5. *If $\varkappa_{\mathscr{R}}\colon \mathscr{R}_1 \to \mathscr{R}_2$ and $\varkappa_{\mathscr{R}}'\colon \mathscr{R}_1' \to \mathscr{R}_2'$ are two t-resolutions of a morphism $\varkappa\colon G_1 \to G_2$, then there exists a third t-resolution $\varkappa_{\mathscr{R}}''$ of $\varkappa$ which dominates both $\varkappa_{\mathscr{R}}$ and $\varkappa_{\mathscr{R}}'$.*

Proof. By Corollary 2.10, there exists a t-resolution $\mathscr{R}_2''$ of G_2 which dominates both $\mathscr{R}_2$ and $\mathscr{R}_2'$. On the other hand, by Lemma 3.3, there exists a t-resolution $\widetilde{\varkappa}_{\mathscr{R}}\colon \mathscr{R}_1''' \to \mathscr{R}_2''$ of $\varkappa$ for a suitable choice of t-resolution $\mathscr{R}_1'''$ of G_1. Now a second application of Corollary 2.10 yields a t-resolution $\mathscr{R}_1''$ of G_1 which dominates $\mathscr{R}_1$, $\mathscr{R}_1'$ and $\mathscr{R}_1'''$. Let $\phi\colon \mathscr{R}_1'' \to \mathscr{R}_1'''$ be the corresponding morphism, which is a t-resolution of id_{G_1}. Then $\varkappa_{\mathscr{R}}'' = \widetilde{\varkappa}_{\mathscr{R}} \circ \phi\colon \mathscr{R}_1'' \to \mathscr{R}_2''$ is a t-resolution of $\varkappa$ which dominates both $\varkappa_{\mathscr{R}}$ and $\varkappa_{\mathscr{R}}'$. $\square$

Construction 3.6. Let $\varkappa\colon G_1 \to G_2$ be a homomorphism of reductive S-group schemes. By Lemma 3.3, there exists a t-resolution $\varkappa_{\mathscr{R}}\colon \mathscr{R}_1 \to \mathscr{R}_2$ of $\varkappa$, which induces a homomorphism $\pi_1(\varkappa_{\mathscr{R}})\colon \pi_1(\mathscr{R}_1) \to \pi_1(\mathscr{R}_2)$ of finitely generated twisted constant S-group schemes. Thus, if we identify $\pi_1(G_i)$ with $\pi_1(\mathscr{R}_i)$ for $i = 1, 2$, we obtain an S-homomorphism $\pi_1(\varkappa_{\mathscr{R}})\colon \pi_1(G_1) \to \pi_1(G_2)$ which, by Lemma 3.5, can be shown to be independent of the chosen t-resolution $\varkappa_{\mathscr{R}}$ of $\varkappa$. We denote it by

$$\pi_1(\varkappa)\colon \pi_1(G_1) \to \pi_1(G_2).$$

Lemma 3.7. *Let $G_1 \xrightarrow{\varkappa} G_2 \xrightarrow{\lambda} G_3$ be homomorphisms of reductive S-group schemes. Then*

$$\pi_1(\lambda \circ \varkappa) = \pi_1(\lambda) \circ \pi_1(\varkappa).$$

Proof. Choose a t-resolution $\mathscr{R}_3$ of G_3. Applying Lemma 3.3 first to λ and then to $\varkappa$, we obtain t-resolutions $\mathscr{R}_1 \xrightarrow{\varkappa_{\mathscr{R}}} \mathscr{R}_2 \xrightarrow{\lambda_{\mathscr{R}}} \mathscr{R}_3$ of $\varkappa$ and λ, and the composition $\lambda_{\mathscr{R}} \circ \varkappa_{\mathscr{R}}$ is a t-resolution of $\lambda \circ \varkappa$. Thus, by Remark 3.2,

$$\pi_1(\lambda \circ \varkappa) = \pi_1(\lambda_{\mathscr{R}} \circ \varkappa_{\mathscr{R}}) = \pi_1(\lambda_{\mathscr{R}}) \circ \pi_1(\varkappa_{\mathscr{R}}) = \pi_1(\lambda) \circ \pi_1(\varkappa),$$

as claimed. □

Summarizing, for any non-empty scheme S, we have constructed a covariant functor π_1 from the category of reductive S-group schemes to the category of finitely generated twisted constant S-group schemes. When S is admissible in the sense of [GA1, Definition 2.1], so that every reductive S-group scheme admits a flasque resolution [GA1, Proposition 3.2], the functor π_1 defined here coincides with the functor π_1 defined in [GA1, Definition 3.7] in terms of flasque resolutions, because a flasque resolution is a particular case of a t-resolution. A basic example of a non-admissible scheme S to which the constructions of the present paper apply (but not those of [GA1]) is an algebraic curve over a field having an ordinary double point. See [GA1, Remark 2.3].

The following result generalizes [BKG, Lemma 3.7], [CT, Proposition 6.8] and [GA1, Theorem 3.14].

Theorem 3.8. *Let $1 \to G_1 \to G_2 \to G_3 \to 1$ be an exact sequence of reductive S-group schemes. Then the induced sequence of finitely generated twisted constant S-group schemes*

$$0 \to \pi_1(G_1) \to \pi_1(G_2) \to \pi_1(G_3) \to 0$$

is exact.

Proof. The proof is similar to the proof of [GA1, Theorem 3.14] using (13) above. Namely, one first proves the theorem when G_1 is semisimple using the same arguments as in the proof of [GA1, Lemma 3.12] (those arguments rely on [GA1, Proposition 2.8], which is valid over any non-empty base scheme S). Secondly, one proves the theorem when G_1 is an S-torus using the same arguments as in the proof of [GA1, Lemma 3.13] (which rely on [GA1, Proposition 2.9], which again holds over any non-empty base scheme S). Finally, the theorem is obtained by combining these two particular cases as in the proof of [GA1, Theorem 3.14]. □

We will now present a second proof of Theorem 3.8 which relies on the étale-local existence of maximal tori in reductive S-group schemes. To this end, we will first show that if G is a reductive S-group scheme which contains a maximal torus T, then T canonically determines a t-resolution of G.

Lemma 3.9. *Let G be a reductive S-group scheme having a maximal S-torus T and let $\widetilde{T} := \widetilde{G} \times_G T$, which is a maximal S-torus in $\widetilde{G}$. Then there exists a t-resolution of G*

$$1 \to \widetilde{T} \to H \to G \to 1 \qquad (\mathscr{R}_T)$$

such that $H^{\rm tor}$ is canonically isomorphic to T.

Proof. By [GA1, proof of Proposition 3.2], the product in G and the canonical epimorphism $\widetilde{G}\to G^{\rm der}$ induce a faithfully flat homomorphism $\mathrm{rad}(G)\times_S \widetilde{G}\to G$ whose (central) kernel μ_1 embeds into $Z(\widetilde{G}\,)$ via the canonical projection $\mathrm{rad}(G)\times_S\widetilde{G}\to\widetilde{G}$. In particular, we have a central extension

$$1\to\mu_1\overset{\varphi}{\to}\mathrm{rad}(G)\times_S\widetilde{G}\to G\to 1. \tag{14}$$

Since $Z(\widetilde{G}\,)\subset\widetilde{T}$ by [SGA3$_{\rm new}$, Exp. XXII, Corollary 4.1.7], we obtain an embedding $\psi\colon\mu_1\hookrightarrow\widetilde{T}$. Let H be the pushout of $\varphi\colon\mu_1\hookrightarrow\mathrm{rad}(G)\times_S\widetilde{G}$ and $\psi\colon\mu_1\hookrightarrow\widetilde{T}$, i.e., the cokernel of the central embedding

$$(\varphi,\mathrm{inv}_{\widetilde{T}}\circ\psi)_S\colon\mu_1\hookrightarrow\big(\mathrm{rad}(G)\times_S\widetilde{G}\big)\times_S\widetilde{T}. \tag{15}$$

Now let $\varepsilon\colon S\to\mathrm{rad}(G)\times_S\widetilde{G}$ be the unit section of $\mathrm{rad}(G)\times_S\widetilde{G}$ and set

$$j=(\varepsilon,\mathrm{id}_{\widetilde{T}})_S\colon S\times_S\widetilde{T}\to\big(\mathrm{rad}(G)\times_S\widetilde{G}\big)\times_S\widetilde{T}.$$

Composing j with the canonical isomorphism $\widetilde{T}\simeq S\times_S\widetilde{T}$, we obtain an S-morphism $\widetilde{T}\to\mathrm{rad}(G)\times_S\widetilde{G}\times_S\widetilde{T}$ which induces an embedding $\iota_T\colon\widetilde{T}\hookrightarrow H$. Further, let $\pi_T\colon H\to G$ be the homomorphism which is induced by the projection

$$\mathrm{rad}(G)\times_S\widetilde{G}\times_S\widetilde{T}\to\mathrm{rad}(G)\times_S\widetilde{G}.$$

Then we obtain a t-resolution of G

$$1\longrightarrow\widetilde{T}\overset{\iota_T}{\longrightarrow}H\overset{\pi_T}{\longrightarrow}G\longrightarrow 1$$

which is canonically determined by T (cf. the proof of Proposition 2.3). It remains to show that $H^{\rm tor}$ is canonically isomorphic to T. Let $\varepsilon_{\rm rad}\colon S\to\mathrm{rad}(G)$ and $\varepsilon_{\widetilde{T}}\colon S\to\widetilde{T}$ be the unit sections of $\mathrm{rad}(G)$ and $\widetilde{T}$, respectively, and consider the homomorphism

$$(\varepsilon_{\rm rad},\mathrm{id}_{\widetilde{G}},\varepsilon_{\widetilde{T}})_S\colon S\times_S\widetilde{G}\times_S S\to\mathrm{rad}(G)\times_S\widetilde{G}\times_S\widetilde{T}.$$

Composing the above homomorphism with the canonical isomorphism $\widetilde{G}\simeq S\times_S\widetilde{G}\times_S S$, we obtain a canonical embedding $\widetilde{G}\hookrightarrow\mathrm{rad}(G)\times_S\widetilde{G}\times_S\widetilde{T}$. The latter map induces a homomorphism $\widetilde{G}\to H$ which identifies $\widetilde{G}$ with $H^{\rm der}$. Now consider the composite homomorphism

$$\varphi_{\rm rad}\colon\mu_1\overset{\varphi}{\longrightarrow}\mathrm{rad}(G)\times_S\widetilde{G}\overset{\mathrm{pr}_1}{\longrightarrow}\mathrm{rad}(G).$$

Then $H^{\rm tor}:=H/H^{\rm der}=H/\widetilde{G}$ is isomorphic to the cokernel of the central embedding

$$(\varphi_{\rm rad},\mathrm{inv}_{\widetilde{T}}\circ\psi)_S\colon\mu_1\hookrightarrow\mathrm{rad}(G)\times_S\widetilde{T}. \tag{16}$$

Compare (15). Finally, the canonical embedding $\widetilde{T}\hookrightarrow\widetilde{G}$ induces an embedding $H^{\rm tor}\hookrightarrow G$ (see (14) and (16)) whose image is $\mathrm{rad}(G)\cdot(T\cap G^{\rm der})=T$ [SGA3$_{\rm new}$, Exp. XXII, proof of Proposition 6.2.8(i)]. This completes the proof. $\square$

Remark 3.10. It is clear from the above proof that the homomorphism $\widetilde{T} \to H^{\mathrm{tor}} = T$ induced by the t-resolution $\mathscr{R}_T$ of Lemma 3.9 is the canonical homomorphism $\partial\colon \widetilde{T} \to T$.

Definition 3.11. Let G be a reductive S-group scheme containing a maximal S-torus T. The *algebraic fundamental group of the pair* (G, T) is the S-group scheme $\pi_1(G, T) := \mathrm{Coker}\,[\partial_*\colon \widetilde{T}_* \to T_*]$.

By Lemma 3.9 and Definition 2.12 we have a canonical isomorphism

$$\vartheta_T\colon \pi_1(G, T) \xrightarrow{\sim} \pi_1(\mathscr{R}_T) = \pi_1(G). \tag{17}$$

Further, any morphism of pairs $\varkappa\colon (G_1, T_1) \to (G_2, T_2)$ (in the obvious sense) induces an S-homomorphism $\varkappa_*\colon \pi_1(G_1, T_1) \to \pi_1(G_2, T_2)$. It can be shown that the following diagram commutes:

$$\begin{tikzcd} \pi_1(G_1, T_1) \arrow[r, "\varkappa_*"] \arrow[d, "\vartheta_{T_1}"'] & \pi_1(G_2, T_2) \arrow[d, "\vartheta_{T_2}"] \\ \pi_1(G_1) \arrow[r, "\varkappa_*"] & \pi_1(G_2)\,. \end{tikzcd} \tag{18}$$

This is immediate in the case where $\varkappa$ is a *normal* homomorphism, i.e. $\varkappa(G_1)$ is normal in G_2 (this is the only case needed in this paper). Indeed, in this case we have $\varkappa(\mathrm{rad}(G_1)) \subset \mathrm{rad}(G_2)$ and therefore $\varkappa$ induces a morphism of t-resolutions $\varkappa_{\mathscr{R}}\colon \mathscr{R}_{T_1} \to \mathscr{R}_{T_2}$. See the proof of Lemma 3.9.

Remark 3.12. The preceding considerations and Lemma 2.11 show that, if S is an admissible scheme in the sense of [GA1, Definition 2.1], so that every reductive S-group scheme G admits a flasque resolution $\mathscr{F}$, and G contains a maximal S-torus T, then there exists a canonical isomorphism $\pi_1(\mathscr{F}) \cong \mathrm{Coker}\,[\partial_*\colon \widetilde{T}_* \to T_*]$. This fact generalizes [CT, Proposition A.2], which is the case $S = \mathrm{Spec}\, k$, where k is a field, of the present remark.

Lemma 3.13. *Let*

$$1 \to (G_1, T_1) \xrightarrow{\varkappa} (G_2, T_2) \xrightarrow{\lambda} (G_3, T_3) \to 1$$

be an exact sequence of reductive S-group schemes with maximal tori. Then the sequence of étale, finitely generated twisted constant S-group schemes

$$0 \to \pi_1(G_1, T_1) \xrightarrow{\varkappa_*} \pi_1(G_2, T_2) \xrightarrow{\lambda_*} \pi_1(G_3, T_3) \to 0$$

is exact.

Proof. The assertion of the lemma is local for the étale topology, so we may and do assume that T_1, T_2, and T_3 are split. By [GA1, Proposition 2.10], there exists an exact commutative diagram of reductive S-group schemes

$$\begin{tikzcd} 1 \arrow[r] & \widetilde{G}_1 \arrow[r] \arrow[d, "\partial_1"] & \widetilde{G}_2 \arrow[r] \arrow[d, "\partial_2"] & \widetilde{G}_3 \arrow[r] \arrow[d, "\partial_3"] & 1 \\ 1 \arrow[r] & G_1 \arrow[r] & G_2 \arrow[r] & G_3 \arrow[r] & 1, \end{tikzcd}$$

which induces an exact commutative diagram of split S-tori

$$\begin{tikzcd} 1 \arrow[r] & \widetilde{T}_1 \arrow[r] \arrow[d, "\partial_1"] & \widetilde{T}_2 \arrow[r] \arrow[d, "\partial_2"] & \widetilde{T}_3 \arrow[r] \arrow[d, "\partial_3"] & 1 \\ 1 \arrow[r] & T_1 \arrow[r] & T_2 \arrow[r] & T_3 \arrow[r] & 1, \end{tikzcd} \tag{19}$$

where $\widetilde{T}_i := \widetilde{G}_i \times_{G_i} T_i$ $(i = 1, 2, 3)$. Now, as in [BKG, Proof of Lemma 3.7], diagram (19) induces an exact commutative diagram of constant S-group schemes

$$\begin{tikzcd} 1 \arrow[r] & \widetilde{T}_{1*} \arrow[r] \arrow[d, hook, "\partial_{1*}"] & \widetilde{T}_{2*} \arrow[r] \arrow[d, hook, "\partial_{2*}"] & \widetilde{T}_{3*} \arrow[r] \arrow[d, hook, "\partial_{3*}"] & 1 \\ 1 \arrow[r] & T_{1*} \arrow[r] & T_{2*} \arrow[r] & T_{3*} \arrow[r] & 1 \end{tikzcd}$$

with injective vertical arrows. An application of the snake lemma to the last diagram now yields the exact sequence

$$0 \to \mathrm{Coker}\ \partial_{1*} \to \mathrm{Coker}\ \partial_{2*} \to \mathrm{Coker}\ \partial_{3*} \to 0,$$

which is the assertion of the lemma. □

Second proof of Theorem 3.8. Let $1 \to G_1 \to G_2 \to G_3 \to 1$ be an exact sequence of reductive S-group schemes. By [SGA3$_{\text{new}}$, Exp. XIX, Proposition 6.1], for any reductive S-group scheme G there exists an étale covering $\{S_\alpha \to S\}_{\alpha\in A}$ such that each $G_{S_\alpha} := G \times_S S_\alpha$ contains a split maximal S_α-torus T_α. Thus, since the assertion of the theorem is local for the étale topology, we may and do assume that G_2 contains a split maximal S-torus T_2. Let $T_1 = G_1 \times_{G_2} T_2$ and let T_3 be the cokernel of $T_1 \to T_2$. Then T_i is a split maximal S-torus in G_i for $i = 1, 2, 3$ and we have an exact sequence of pairs

$$1 \to (G_1, T_1) \to (G_2, T_2) \to (G_3, T_3) \to 1.$$

Now the theorem follows from Lemma 3.13, (17) and (18). □

4. Abelian cohomology and t-resolutions

Let S_{fl} (respectively, $S_{\text{ét}}$) be the small fppf (respectively, étale) site over S. If F_1 and F_2 are abelian sheaves on S_{fl} (regarded as complexes concentrated in degree 0), $F_1 \otimes^{\mathbf{L}} F_2$ (respectively, $\mathrm{RHom}\,(F_1, F_2)$) will denote the total tensor product (respectively, right derived Hom functor) of F_1 and F_2 in the derived category of the category of abelian sheaves on S_{fl}.

Let G be a reductive group scheme over S. For any integer $i \geq -1$, the i-th *abelian (flat) cohomology group of* G is by definition the hypercohomology group

$$H^i_{\text{ab}}(S_{\text{fl}}, G) = \mathbb{H}^i\big(S_{\text{fl}}, Z\big(\widetilde{G}\,\big) \xrightarrow{\partial_Z} Z(G)\big).$$

On the other hand, the i-th *dual abelian cohomology group of* G is the group

$$H^i_{\text{ab}}(S_{\text{ét}}, G^*) = \mathbb{H}^i\big(S_{\text{ét}}, Z(G)^* \xrightarrow{\partial_Z^*} Z\big(\widetilde{G}\,\big)^*\big).$$

Here all the complexes of length 2 are in degrees $(-1, 0)$. See [GA1, beginning of §4] for basic properties of these cohomology groups and [Bo1, GA2, GA3] for (some of) their arithmetical applications.

The following result is an immediate consequence of (5).

Proposition 4.1. *Let G be a reductive S-group scheme and let $1 \to T \to H \to G \to 1$ be a t-resolution of G. Then the given t-resolution defines isomorphisms $H^i_{\text{ab}}(S_{\text{fl}}, G) \simeq \mathbb{H}^i\big(S_{\text{fl}}, T \to R\big)$ and $H^i_{\text{ab}}(S_{\text{ét}}, G^*) \simeq \mathbb{H}^i\big(S_{\text{ét}}, R^* \to T^*\big)$, where $R = H^{\text{tor}}$. Further, there exist exact sequences*

$$\ldots \to H^i(S_{\text{ét}}, T) \to H^i(S_{\text{ét}}, R) \to H^i_{\text{ab}}(S_{\text{fl}}, G) \to H^{i+1}(S_{\text{ét}}, T) \to \ldots$$

and

$$\ldots \to H^i(S_{\text{ét}}, R^*) \to H^i(S_{\text{ét}}, T^*) \to H^i_{\text{ab}}(S_{\text{ét}}, G^*) \to H^{i+1}(S_{\text{ét}}, R^*) \to \ldots. \qquad \square$$

Corollary 4.2. *Let G be a reductive S-group scheme. Then, for every integer $i \geq -1$, there exist isomorphisms*

$$H^i_{\text{ab}}(S_{\text{fl}}, G) \simeq \mathbb{H}^i(S_{\text{fl}}, \pi_1(G) \otimes^{\mathbf{L}} \mathbb{G}_{m,S})$$

and

$$H^i_{\text{ab}}(S_{\text{ét}}, G^*) \simeq \mathbb{H}^i(S_{\text{ét}}, \mathrm{RHom}(\pi_1(G), \mathbb{Z}_S)).$$

Proof. This follows from Proposition 4.1 in the same way as [GA1, Corollary 4.3] follows from [GA1, Proposition 4.2]. $\square$

Proposition 4.3. *Let $1 \to G_1 \to G_2 \to G_3 \to 1$ be an exact sequence of reductive S-group schemes. Then there exist exact sequences of abelian groups*

$$\ldots \to H^i_{\text{ab}}(S_{\text{fl}}, G_1) \to H^i_{\text{ab}}(S_{\text{fl}}, G_2) \to H^i_{\text{ab}}(S_{\text{fl}}, G_3) \to H^{i+1}_{\text{ab}}(S_{\text{fl}}, G_1) \to \ldots$$

and

$$\ldots \to H^i_{\text{ab}}(S_{\text{ét}}, G_3^*) \to H^i_{\text{ab}}(S_{\text{ét}}, G_2^*) \to H^i_{\text{ab}}(S_{\text{ét}}, G_1^*) \to H^{i+1}_{\text{ab}}(S_{\text{ét}}, G_3^*) \to \ldots.$$

Proof. This follows from Corollary 4.2 and Theorem 3.8. $\square$

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References

[Bo1] Borovoi, M.: *Abelian Galois cohomology of reductive groups,* Mem. Amer. Math. Soc. **132** (1998), no. 626.

[Bo2] Borovoi, M.: *The algebraic fundamental group of a reductive group scheme over an arbitrary base scheme,* Appendix B in `arXiv:1112.6020v1[math.NT]`.

[BKG] Borovoi, M., Kunyavskiĭ B. and Gille, P.: *Arithmetical birational invariants of linear algebraic groups over two-dimensional geometric fields,* J. Algebra **276** (2004), 292–339.

[CT] Colliot-Thélène, J.-L.: *Résolutions flasques des groupes linéaires connexes.* J. reine angew. Math. **618** (2008), 77–133.

[Co] Conrad, B.: *Reductive group schemes (SGA3 Summer School, 2011),* Available at `http://math.stanford.edu/~conrad/papers/luminysga3.pdf`.

[$\mathrm{SGA3}_{\mathrm{new}}$] Demazure, M. and Grothendieck, A. (Eds.): *Schémas en groupes.* Séminaire de Géométrie Algébrique du Bois Marie 1962-64 (SGA 3). Augmented and corrected 2008-2011 re-edition of the original by P. Gille and P. Polo. Available at `http://www.math.jussieu.fr/~polo/SGA3`. Volumes 1 and 3 are now published: Documents mathématiques **7** (2011) and **8** (2011), Société Mathématique de France.

[GM] Gelfand, S.I.; Manin, Yu.I.: *Methods of homological algebra,* 2nd ed., Springer-Verlag, Berlin, 2003.

[GA1] González-Avilés, C.D.: *Flasque resolutions of reductive group schemes,* To appear in Cent. Eur. J. Math. `arXiv:1112.6020v2[math.NT]`.

[GA2] González-Avilés, C.D.: *Quasi-abelian crossed modules and nonabelian cohomology.* J. Algebra **369** (2012), 235–255.

[GA3] González-Avilés, C.D.: *Abelian class groups of reductive group schemes.* To appear in Israel J. Math. `arXiv:1108.3264v4[math.NT]`.

[Ko] Kottwitz, R.E.: *Stable trace formula: Cuspidal tempered terms,* Duke Math. J. **51** (1984), 611–650.

[Me] Merkurjev, A.S.: *K-theory and algebraic groups*, European Congress of Mathematics, Vol. II (Budapest, 1996), pp. 43–72, Progr. Math., 169, Birkhäuser, Basel, 1998.

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